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THE DISTURBANCE DUE TO PLANE AND LINE SOURCES IN A PRE-STRESSED SEMI-INFINITE ELASTIC SOLID—I

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Abstract—This paper investigates the disturbance due to harmonic time dependence plane and line load in pre-stressed elastic half-space. A normal force has been considered to act on the surface of a semi-infinite elastic solid. Stresses and displacements due to the load are evaluated. The results obtained here are in agreement with the classical Lamb's problem in elastic wave propagation when the half-space is supposed to be initially stress-free. The numerical results include the stress distribution and the displacement at the surface; further, the results for stresses are shown graphically.

INTRODUCTION

One of the contributions of lasting significance in the area of wave propagation in elastic solids is a great classical paper in seismology by Lamb[1]. In this work, Lamb investigated the wave motion generated at the surface of an elastic half-space by the application of concentrated loads at the surface or inside the half-space. He also discussed the case of surface motions generated by a line load and a point load applied normally to the surface. Both the harmonic time dependent and impulsive loads were considered. In recent years the methods and solutions in Lamb's paper have been cast in a somewhat more elegant form and more detailed computations have been carried out by Pekeris[2] and Garvin[3], particularly for loads of arbitrary time dependence.

The earth is considered to be an initially stressed body. Due to atmospheric pressure, gravity variation, creep, etc., enormous initial stresses may exist inside the earth. Therefore, it is of interest to study the problem of earth models with initial stresses, though little attention has been given by the previous authors. Biot[4] mentioned that initial stresses has a remarkable effect on the propagation of elastic waves in body. He showed that the propagation of waves in an initially stressed body is fundamentally different from the classical theory of elasticity to a great extent.

The present problem deals with the determination of stresses and displacements in an initially stressed elastic semi-infinite medium under the action of a normal force (assumed as a plane and line sources). It is of interest to determine the stresses and displacements developed due to the load in an isotropic solid under pre-stressed conditions, showing the influence of the pre-stress on the propagation of wave. Displacement components have been obtained by contour integration[5]. The results obtained in this paper are exactly the same as given in [5], when the half-space is initially stress free. For different values of initial stresses, the numerical results for stresses and displacements at the surface have been computed and the results for the stresses are presented in the form of graphs.

FORMULATION OF THE PROBLEM

Let the homogeneous half-space $y \geq 0$ be under uniform initial compressive stresses S_{11} and S_{22} acting along x- and y-directions respectively (Fig. 1). Consider the propagation of elastic waves, produced by the action of a normal force (assumed as a plane and line sources) which vary harmonically in time. In the xy-plane, there arise both compressional and shear waves and the resulting displacements may be expressed in terms of two scalar functions ϕ and ψ each of which depend on x, y and t.

FIELD EQUATIONS FOR A PRE-STRESSED ELASTIC HALF-SPACE

A general deformation of a pre-stressed elastic body consists of a pure deformation

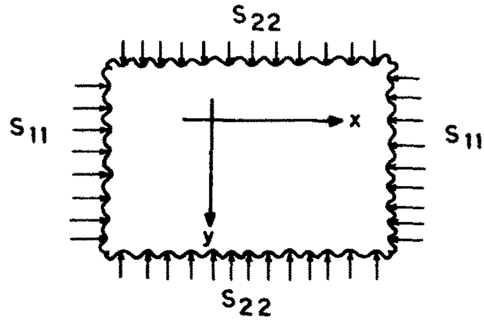

Fig. 1. Model considered.

followed by a rigid body rotation. If the pre-stressed body is given a small general deformation, some additional stresses may develop inside the body. These additional stresses may be classified into two types, viz. (i) stresses due to rotation only which is of purely geometric origin (ii) stresses due to pure deformation which is of physical nature.

Due to rigid body rotation the initial axes of reference will be rotated. If the additional stress system is calculated with reference to original axes, these stresses will contain (*a*) and (*b*) both and if they are evaluated with respect to rotated axes, the stress system will be due to pure deformation only.

Biot[6] has established the relations between (*a*) and (*b*), and has termed (*b*) incremental stresses s_{ij} $(i = 1, 2;\ j = 1, 2)$ over initial stresses S_{ij}. The strain components e_{ij} have also been calculated with respect to rotated axes.

The dynamical equations of equilibrium for a pre-stressed elastic body are [6]

$$\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - P\frac{\partial \omega}{\partial y} = \rho\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P\frac{\partial \omega}{\partial x} = \rho\frac{\partial^2 v}{\partial t^2} \tag{1}$$

where $P = S_{22} - S_{11}$; further, ω is the rotational component, i.e. $\omega = 1/2(\partial v/\partial x - \partial u/\partial y)$, in which u and v are the displacements along x- and y-directions respectively. Further, the incremental stress–strain relations are [6]

$$s_{11} = (\lambda + 2\mu + P)e_{xx} + (\lambda + P)e_{yy}$$

$$s_{22} = \lambda e_{xx} + (\lambda + 2\mu)e_{yy}$$

$$s_{12} = 2\mu e_{xy} \tag{2}$$

where λ, μ are Lame's constants and the incremental strain tensor (e_{ij}) may be defined as

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad (i = j = 1, 2). \tag{2a}$$

The equations of motion (1), with the help of (2) and (2a), may be expressed in terms of displacement components as

$$(\lambda + 2\mu + P)\frac{\partial^2 u}{\partial x^2} + \left(\lambda + \mu + \frac{P}{2}\right)\frac{\partial^2 v}{\partial x \partial y} + \left(\mu + \frac{P}{2}\right)\frac{\partial^2 U}{\partial y^2} = \rho\frac{\partial^2 u}{\partial t^2}$$

$$(\lambda + 2\mu)\frac{\partial^2 v}{\partial y^2} + \left(\lambda + \mu + \frac{P}{2}\right)\frac{\partial^2 u}{\partial x \partial y} + \left(\mu - \frac{P}{2}\right)\frac{\partial^2 v}{\partial x^2} = \rho\frac{\partial^2 v}{\partial t^2}. \tag{3}$$

Further, specifying the displacement components in terms of the wave potentials ϕ and ψ as

$$u = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x} \tag{4}$$

it may be shown that eqns (3) are satisfied if the functions ϕ and ψ are solutions of equations

$$\nabla^2\phi = \frac{\rho}{\lambda + 2\mu + P}\frac{\partial^2\phi}{\partial t^2}, \qquad \nabla^2\psi = \frac{\rho}{\mu = \frac{P}{2}}\frac{\partial^2\psi}{\partial t^2} \tag{5a, b}$$

and

$$\nabla^2\phi = \frac{\rho}{\lambda + 2\mu}\frac{\partial^2\phi}{\partial t^2}, \qquad \nabla^2\psi = \frac{\rho}{\mu - \frac{P}{2}}\frac{\partial^2\psi}{\partial t^2}. \tag{5c, d}$$

Equations (5a, c) represent compressional waves along the x- and y-axes respectively, and eqns (5b, d) represent distortional waves along the same directions. For example, the displacements corresponding to compressional waves are

$$u = u_0' \cos(k'x - \alpha t)$$
$$v = 0$$

The first equation of (3) yields

$$(\lambda + 2\mu + P)\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}$$

showing that the velocity of compressional wave along the x-direction is

$$\sqrt{\left(\frac{\lambda + 2\mu + P}{\rho}\right)}.$$

Similarly, the displacements due to the distortional waves are

$$u = 0$$
$$v = v_0' \cos(k'x - \alpha t).$$

The second equation of equations of motion (3) reduces to

$$\left(\mu - \frac{P}{2}\right)\frac{\partial^2 v}{\partial x^2} = \rho\frac{\partial^2 v}{\partial t^2}$$

and the velocity of distortional waves along the x-direction is

$$\sqrt{\left(\frac{\mu - P/2}{\rho}\right)}.$$

These results bring out the important fact that the propagation of elastic waves in an initially stressed body is fundamentally different from the stress-free case and cannot be represented by simply introducing into the classical theory the stress-dependent elastic coefficients [4]. In a similar way it may be shown that eqns (5b, c) represent distortional and compressional waves respectively along the y-direction.

The compressional and distortional waves along the x-axis only will be considered here.

SOLUTION

Assume simple harmonic motion with one and the same frequency

$$\phi = e^{ipt}, \qquad \psi \sim e^{ipt}$$

$$\frac{\partial^2 \phi}{\partial t^2} = -p^2\phi.$$

The eqns (5a) and (5d) become

$$(\nabla^2 + h^2)\phi = 0, \qquad (\nabla^2 + k^2)\psi = 0 \tag{6a}$$

where

$$h^2 = \frac{p^2\rho}{\lambda + 2\mu + P}, \qquad k^2 = \frac{p^2\rho}{\mu - P/2}. \tag{6b}$$

Consider the sources localized to the plane $y = 0$ (Fig. 2). Omitting the time factor e^{ipt} the solutions for y 0 are

$$\varphi = A\,e^{-\alpha y}\,e^{i\xi x}, \qquad \psi = B\,e^{-\beta y}\,e^{i\xi x} \tag{7}$$

where ξ is wave number in x-direction and is real. Further, from (6) and (7)

$$\frac{\alpha^2}{\xi^2} = 1 - \frac{c^2}{\alpha_1^2}, \qquad \frac{\beta^2}{\xi^2} = 1 - \frac{c^2}{\beta_1^2}, \tag{7a}$$

in which $c = (p/\xi)$, velocity of wave due to vibration in the medium; $\alpha_1^2 = (\lambda + 2\mu + P/\rho)$, velocity of P-wave in the medium under initial stress; and $\beta_1^2 = (\mu - P/2)/(\rho)$, velocity of S-wave in the medium under initial stress.

Boundary conditions

Let a sinusoidal normal force $Y\,e^{i\xi x}$ be applied on the plane $y = 0$, i.e. the surface of the pre-stressed elastic solid (Fig. 3). The boundary conditions may be written as

$$\Delta f_x = 0, \qquad \Delta f_y = Y\,e^{i\xi x} \tag{8}$$

where Δf_x and Δf_y are the incremental boundary forces per unit initial area and these are given by

$$\Delta f_x = s_{12} - S_{22} - S_{11}e_{xy}, \qquad \Delta f_y = s_{22} + S_{22}e_{xr}. \tag{9}$$

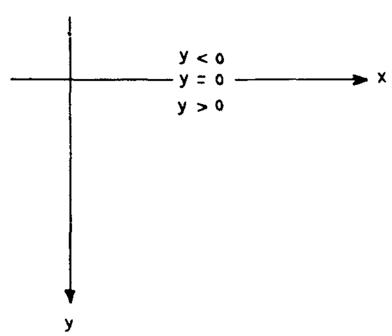

Fig. 2. Sources at the plane $y = 0$.

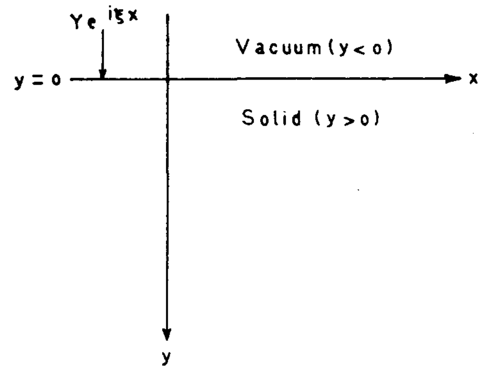

Fig. 3. Normal force acting on the plane $y = 0$.

Stresses and displacements in the pre-stressed elastic half-space

The displacements and the stresses at $y = 0$ may be obtained as displacements at $y = 0$:

$$\begin{aligned} &\text{tangential} \qquad u_0 = (i\xi A - \beta B)\, e^{i\xi x} \\ &\text{normal} \qquad v_0 = (-\alpha A - i\xi B)\, e^{i\xi x}. \end{aligned} \tag{10}$$

Incremental stresses at $y = 0$:

$$\begin{aligned} &\text{tangential} \qquad [s_{12}]_{y=0} = \mu[-2i\xi\alpha A + (\xi^2 + \beta^2)B]\, e^{i\xi x} \\ &\text{normal} \qquad [s_{11}]_{y=0} = [\{(\lambda + P)(\alpha^2 - \xi^2) - 2\mu\xi^2\}A - 2i\mu\xi\beta B]\, e^{i\xi x} \\ &\qquad [s_{22}]_{y=0} = [\{\lambda(\alpha^2 - \xi^2) + 2\mu\alpha^2\}A + 2i\mu\xi\beta B]\, e^{i\xi x}. \end{aligned} \tag{11}$$

By means of (9)–(11), the boundary conditions (8) yield $2i\xi\alpha(2\mu - S_{11})A - [(\xi^2 + \beta^2)(2\mu - S_{11}) + S_{22}(\beta^2 - \xi^2)]B = 0$

$$[(\lambda + 2\mu)\alpha^2 - (\lambda + S_{22})\xi^2]A + (2\mu - S_{22})i\xi\beta B = Y,$$

from which the potentials are determined as

$$\begin{aligned} \phi &= \frac{(2\mu - S_{11})(\xi^2 + \beta^2) + (\beta^2 - \xi^2)S_{22}}{S_{11}M + S_{22}M^* + 2\mu^2 N} Y\, e^{-\alpha y}\, e^{i\xi x} \\ \psi &= \frac{2i(2\mu - S_{11})\xi\alpha}{S_{11}M + S_{22}M^* + 2\mu^2 N} Y\, e^{-\beta y}\, e^{i\xi x} \end{aligned} \tag{12}$$

(considering $\lambda = \mu$)

where

$$\begin{aligned} M &= (\xi^2 + \beta^2)[\xi^2 S_{22} - \mu(3\alpha^2 - \xi^2)] + 2(2\mu - S_{22})\xi^2\alpha\beta \\ M^* &= (\xi^2 - \beta^2)[\xi_2 S_{22} - \mu(3\alpha^2 - \xi^2)] + (2\mu - S_{11})\xi^2[2\alpha\beta - (\xi^2 + \beta^2)] \\ N &= (\xi_2 + \beta^2)(3\alpha^2 - \xi^2) - 4\xi^2\alpha\beta. \end{aligned} \tag{12a}$$

The displacements due to the normal force $Y\, e^{i\xi}{}_x$ are

$$\begin{aligned} u &= \frac{i\xi[\{(2\mu - S_{11})(\xi^2 + \beta^2) + (\beta^2 - \xi^2)S_{22}\}\, e^{-\alpha y} - 2(2\mu - S_{11})\alpha\beta\, e^{-\beta y}]}{S_{11}M + S_{22}M^* + 2\mu^2 N} Y\, e^{i\xi x} \\ v &= \frac{2(2\mu - S_{11})\xi^2\alpha\, e^{-\beta y} - \alpha\{(2\mu - S_{11})(\xi^2 + \beta^2) + (\beta^2 - \xi^2)S_{22}\}\, e^{-\alpha y}}{S_{11}M + S_{22}M^* + 2\mu^2 N}. \end{aligned} \tag{13}$$

The stresses, in the plane $y = 0$, may be written in non-dimensional coefficients as

$$s_{11} = \frac{[E^* + (\eta_2 - \eta_1)(\beta_1/\alpha_1)^2][B^*(1-\eta_1) - \eta_2] - 2C^*D^*(1-\eta_1)(\beta_1/c)^2}{\Delta^*} Y \,\mathrm{e}^{i\xi x}$$

$$s_{22} = \frac{A^*\eta_2 - B^*(1-\eta_1) + 2C^*D^*(1-\eta_1)(\beta_1/c)^2}{\Delta^*} Y \,\mathrm{e}^{i\xi x}$$

$$s_{12} = -\frac{1}{2}\frac{B^*[(1-\eta_1)+\eta_2](\beta_1/c)D^*}{\Delta^*} Y \,\mathrm{e}^{i\xi x} \tag{14}$$

where

$$\Delta^* = \left[B^*\left\{(\beta_1/c)^2\eta_2 - \frac{A^*}{2}\right\} - 2C^*D^*(1-\eta_2)(\beta_1/c)^2\right]\eta_1 + \left[(\beta_1/c)^2 - \frac{A^*}{2} - (2C^*D^* - B^*)(\beta_1/c)^2\right]\eta_2 + \frac{1}{2}[A^*B^* + 4C^*D^*(\beta_1/c)^2] \tag{14a}$$

in which

$$A^* = 2(\beta_1/c) - 3(\beta_1/\alpha_1)^2, \qquad B^* = 2(\beta_1/c)^2 - 1,$$
$$C^* = \sqrt{((\beta_1/c)^2 - 1)}, \qquad D^* = \sqrt{((\beta_1/c)^2 - (\beta_1/\alpha_1)^2)},$$
$$E^* = 2(\beta_1/c)^2 + (\beta_1/\alpha_1)^2, \tag{14b}$$

and

$$\eta_1 = S_{11}/2\mu, \qquad \eta_2 = S_{22}/2\mu \tag{14c}$$

are initial stress parameters. Further, if $\Delta^* = 0$, the eqn (14a) gives the frequency equation of Rayleigh wave in an initially stressed medium.

In the absence of initial stresses, the displacement components in the plane $y = 0$ are given by

$$u_0 = \frac{i\xi(2\xi^2 - k^2 - 2\alpha\beta)\,\mathrm{e}^{i\xi x}}{F(\xi)}\frac{Y}{\mu}; \qquad v_0 = \frac{k^2\alpha\,\mathrm{e}^{i\xi x}}{F(\xi)}\frac{Y}{\mu} \tag{15}$$

where $F(\xi) = (2\xi^2 - k^2)^2 - 4\xi^2\alpha\beta$. Also, $F(\xi) = 0$ is the frequency equation of Rayleigh wave in an initially stress-free medium and the corresponding displacements at $y = 0$ are very high. Relations (15) coincide with the results given in [3].

FORCES ACT ON THE LINE $x = 0$, $y = 0$ ON THE SURFACE OF A SEMI-INFINITE ELASTIC SOLID

Until now, a surface force or area force has been discussed. For further progress of the problem, a line force will be considered. This can be formed in the following way:

The force is compressed so that it acts only on the line $x = 0$, $y = 0$, i.e. on the z-axis. This case may be obtained by superposition of an infinite number of stress distribution with all possible wave numbers ξ. For this purpose, the extraneous pressure may be expressed as

$$\int_{-\infty}^{\infty}\left(-\frac{Y}{2\pi}\right)\mathrm{e}^{i\xi x}\,\mathrm{d}\xi = f(x), \tag{16}$$

say with the inversion formula

$$Y = \int_{-\infty}^{\infty} f(\eta)\,\mathrm{e}^{-i\xi\eta}\,\mathrm{d}\eta.$$

In order to obtain a concentrated force at $x = 0$, it is assumed that the normal force $f(x)$ vanishes everywhere along x-axis except at $x = 0$, where it approaches infinity in such a way that Y is finite. With this assumption, Y is constant.

Introducing (16) and (12), the following wave potentials are obtained

$$\varphi = -\left(\frac{Y}{2\pi}\right)\int_{-\infty}^{\infty}\frac{(2\mu - S_{11})(\xi^2+\beta^2)-(\xi^2-\beta^2)S_{22}}{S_{11}M + S_{22}M^* + 2\mu^2 N}\,\mathrm{e}^{-\alpha y}\,\mathrm{e}^{i\xi x}\,\mathrm{d}\xi$$

$$\psi = -i\left(\frac{Y}{\pi}\right)\int_{-\infty}^{\infty}\frac{(2\mu - S_{11})\alpha\xi}{S_{11}M + S_{22}M^* + 2\mu^2 N}\,\mathrm{e}^{-\beta y}\,\mathrm{e}^{i\xi x}\,\mathrm{d}\xi. \tag{17}$$

The surface displacements can immediately be written as, from (13) and (16)

$$u_0 = -i\left(\frac{Y}{2\pi}\right)\int_{-\infty}^{\infty}\frac{\xi[(2\mu - S_{11})(\xi^2+\beta^2)-(\xi^2-\beta^2)S_{22}-2(2\mu - S_{11})\alpha\beta]}{S_{11}M + S_{22}M^* + 2\mu^2 N}\,\mathrm{e}^{i\xi x}\,\mathrm{d}\xi$$

$$v_0 = -\left(\frac{Y}{2\pi}\right)\int_{-\infty}^{\infty}\frac{\alpha(\xi^2-\beta^2)(2\mu - S_{11} - S_{22})}{S_{11}M + S_{22}M^* + 2\mu^2 N}\,\mathrm{e}^{i\xi x}\,\mathrm{d}\xi.$$

By contour integration the following results for surface displacement due to normal line force are determined

$$\begin{aligned}
u_0 = &-YH\,\mathrm{e}^{i(pt-rx)} + \sqrt{(2/\pi)}(Y/\mu)\frac{[\{3a(\beta_1/\alpha_1)^2 + bq - fd - 1\}r - e]}{[fg - 3j - 2m - 3n(\beta_1/\alpha_1)^2]^2}\cdot\\
&\times\frac{[1-(\beta_1/\alpha_1)^2]^{1/2}}{(c/\beta_1)^{3/2}(\xi x)^{3/2}}\,\mathrm{e}^{i(pt-kx-\pi/4)}\\
&-\sqrt{(2/\pi)}(Y/\mu)\frac{[qt - fd + 2(1-r)f + 5 + \{qt + (1-r)\zeta\}(\alpha_1/\beta_1)^2]r - qe(\alpha_1/\beta_1)^2}{[fd - 3s - m + w(\alpha_1/\beta_1)^2]^2}\\
&\times\frac{[(\alpha_1/\beta_1)^2 - 1]^{1/2}}{(c/\beta_1)^{3/2}(\beta_1/\alpha_1)^{3/2}(\xi x)^{3/2}}\,\mathrm{e}^{i(pt-hx-\pi/4)}
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
v_0 = &-iYK\,\mathrm{e}^{i(pt-rx)} + 2\sqrt{(2/\pi)}(Y/5)\frac{ve}{[fg - 3j - 2m - 3n(\beta_1/\alpha_1)^2]^2}\\
&\times\frac{[1-(\beta_1/\alpha_1)^2]}{(c/\beta_1)^{3/2}(\xi x)^{3/2}}i\,\mathrm{e}^{i(pt-kx-\pi/4)} + \sqrt{(2/\pi)}(Y/2\mu)\frac{\nu}{fd - 3j - 2m - 3n(\beta_1/\alpha_1)^2}\\
&\times\frac{(\alpha_1/\beta_1)^2}{(c/\beta_1)^{3/2}(\beta_1/\alpha_1)^{3/2}(\xi x)^{3/2}}i\,\mathrm{e}^{i(pt-hx-\pi/4)}
\end{aligned}$$

where

$$H = \frac{a'b'c'\{(2b'c' - 2a'^2 + 1)r + q\}}{4R'a'^3b'c' - 2S'a'b'c' - 8T'a\{(a'^2 + c'^2)b'^2 + c'^2a'^2\}} \tag{18a}$$

$$\begin{aligned}
&a = \eta_1 + \eta_2 - 1,\quad b = 4\eta_2 + 7,\quad q = \eta_2,\quad r = 1 - \eta_1,\\
&d = 2\eta_1 + \eta_2 - 1,\quad e = \eta_1 + \eta_2 - \eta_1\eta_2 - 1,\quad f = 1 + 2\eta_2,\\
&g = 3\eta_1 + \eta_2 - 1,\quad j = \eta_1 + 3\eta_2,\quad m = 1 + 2\eta_2^2,\\
&\eta = \eta_1 + \eta_2 + 1,\quad t = 2\eta_2 + 5,\quad \zeta = 1 - 4\eta_2,\\
&s = 2\eta_1 + 3\eta_2,\quad w = \eta_1(2\eta_2 + 1) - \eta_2(2\eta_2 + 3) + 1
\end{aligned} \tag{18b}$$

and

$$K = -\frac{b'^2 c' \nu}{4R'a'^3 b'c' - 2S'a'b'c' - 8T'a'\{(a'^2 + c'^2)b'^2 + c'^2 a'^2\}} \tag{18c}$$

in which

$$\nu = 1 - (\eta_1 - \eta_2)$$

$$a' = \tau/k, \quad b' = \sqrt{((\tau/k)^2 - (h/k)^2)}, \quad c' = \sqrt{((\tau/k)^2 - 1)}$$

$$R' = \mu[(1 + 2\eta_2)(\eta_1 + \eta_2 - 1) - 3(\eta_1 + \eta_2) + \eta_1(1 + 2\eta_2) - (2\eta_2^2 + 3\eta_2) + 3(\eta_2 - \eta_1) + 10]$$

$$S' = \mu[3(\eta_1 + \eta_2)(h/k)^2 - \{\eta_1(1 + 2\eta_2) - \eta_2(2\eta_2 + 3)\} + 2\{1 + 3(h/k)^2\}]$$

$$T' = \mu[\eta_1 + \eta_2 - \eta_1\eta_2 - 1]. \tag{18e}$$

The surface displacement (18) is written as the sum of three terms. The first term is derived from the poles and the other two terms from the branch-line integrals.

The first term in (18) represents surface waves in an initially stressed elastic medium. As the amplitude of the term is independent of x, the surface waves do not suffer attenuation as the distance x from the disturbance increases (in the two-dimensional case); on the other hand, the amplitudes of the remaining terms die out as $1/x^{3/2}$ during the wave propagation. This means that at a large distance from point of application of the load, only the disturbance arriving with the velocity of surface waves will be of appreciable magnitude, i.e. the motion consists essentially of free Rayleigh waves. This is well confirmed by experiments and seismological experience.

In the absence of initial stresses (i.e. $\eta_1 = \eta_2 = 0$), the displacement potentials (18) due to normal line force correspond to [3].

NUMERICAL RESULTS AND DISCUSSION

The aim of the present investigation is to determine the stress distribution and the displacement potentials due to the normal force acting on the surface ($y = 0$), when the medium is initially stressed. The graphs are drawn to show the development of stresses due to the force and the role of initial stresses present in the medium.

For computational work, limestone is considered as an isotropic elastic solid with rigidity = 2.5×10^{11} dynes/cm^2 and density = 2.7 gm/cm^3.

Computation of stresses

Figure 4 gives the measurement of incremental stresses in the body for some fixed values of initial stress parameters. It may be seen that the velocity ratio, β_1/c, is inversely proportional to the incremental stresses (Fig. 4), i.e. the Rayleigh wave speed qualifies the development of incremental stresses inside the elastic solid, except the normal incremental stress (s_{11}) remaining unaffected throughout its propagation. In Fig. 5, the incremental stresses due to the normal force is plotted against the normalized initial stress (η_1) for $\eta 2 = 0.0$. The results clearly indicate that the development of normal incremental stresses is more rapid than shear incremental stresses inside the limestone, which is subjected to higher initial stresses (Fig. 5). Reverse is the case with the Fig. 6, in which higher initial stresses increase the development of normal incremental stresses (along the y-direction) and shear incremental stresses, while the same reduces the normal incremental stresses (along the x-direction).

Computation of displacements

In the equations (18a, c), τ/k is the root of

$$\left(\frac{\tau}{k}\right)^8 - \frac{2[R'S' - 8T'^2\{1 + (h/k)^2\}]}{R'^2 - 16T'^2}\left(\frac{\tau}{k}\right) - \frac{6R'Q' - S'^2 + 16T'^2(h/k)^2}{R'^2 - 16T'^2}\left(\frac{\tau}{k}\right)^4 + \frac{6S'Q'}{R'^2 - 16T'^2}\left(\frac{\tau}{k}\right)^2 + \frac{9Q'^2}{R'^2 - 16T'^2} = 0 \tag{19}$$

(the Rayleigh wave equation in an initially stressed incompressible elastic medium) in which

$$Q' = 3\mu(\eta_1 - \eta_2 - 1)(h/k)^2. \tag{19a}$$

Equation (19) has a root 0.6 when $\eta_1 = 0.40$ and $\eta_2 = 0.20$. Using the value ($= 0.6$), the displacement terms in eqn (18) have been calculated, by taking $\lambda = \mu$; thus, the computational results for the horizontal and vertical displacements, due to the normal (Y) force, at the surface $y = 0$ which is under initial compressive stress parameters of magnitude equal to 0.40 and 0.20, are

$$u_0 = -\frac{Y}{\mu}\left[1.185\frac{\cos(pt-\tau x)}{(\xi x)^{3/2}} - 0.002\frac{\cos(pt-kx-\pi/4)}{(\xi x)^{3/2}} - 0.288\frac{\cos(pt-hx-\pi/4)}{(\xi x)^{3/2}}\right]$$

$$v_0 = -\frac{Y}{\mu}\left[0.265\frac{\sin(pt-\tau x)}{(\xi x)^{3/2}} - 0.003\frac{\sin(pt-kx-\pi/4)}{(\xi x)^{3/2}} - 0.129\frac{\sin(pt-hx-\pi/4)}{(\xi x)^{3/2}}\right]. \tag{20}$$

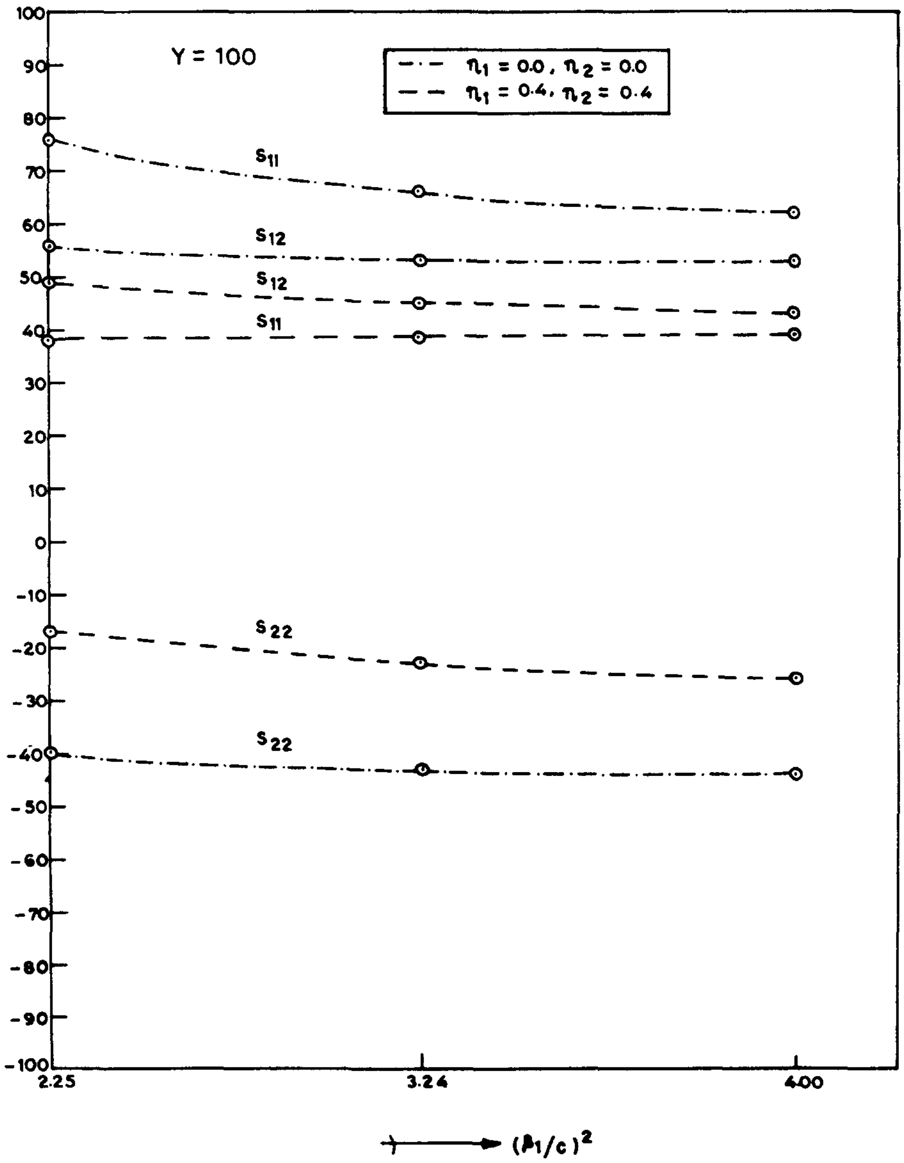

Fig. 4. Variation of incremental stresses with $(\beta_1/c)^2$ for fixed initial stress parameters (η_1 and η_2).

In the absence of initial stresses, eqn (19) has a root equal to 0.95 and corresponding to this value the horizontal and vertical displacements are calculated as

$$u_0 = \frac{Y}{\mu}\left[(0.032+0.010i)\frac{\cos(pt-\tau x)}{(\xi x)^{3/2}} - 0.023\frac{\cos(pt-kx-\pi/4)}{(\xi x)^{3/2}} + 5.443\frac{\cos(pt-hx-\pi/4)}{(\xi x)^{3/2}}\right] \tag{21}$$

$$v_0 = -\frac{Y}{\mu}\left[(0.026+0.012i)\frac{\sin(pt-\tau x)}{(\xi x)^{3/2}} - 0.022\frac{\sin(pt-kx-\pi/4)}{(\xi x)^{3/2}} + 0.962\frac{\sin(pt-hx-\pi/4)}{(\xi x)^{3/2}}\right].$$

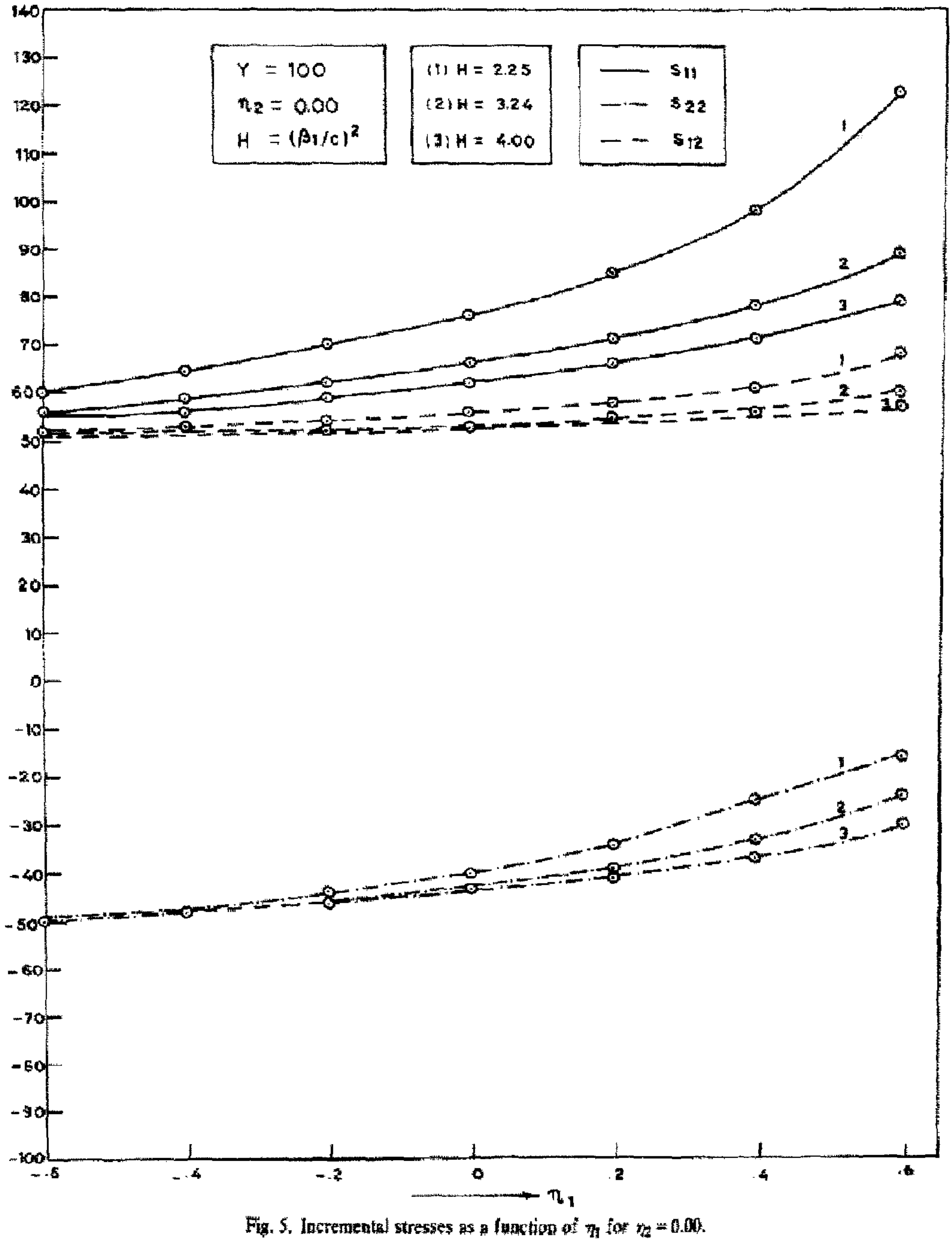

Fig. 5. Incremental stresses as a function of η_1 for $\eta_2 = 0.00$.

Comparing the eqns (21) with (20), it may be seen that the first terms of the former (eqn 21) contain complex values but the same of the latter equation is real-valued. Further, the displacement potentials generated in the body are affected by the presence of the initial stresses in the body.

CONCLUSION

The study of the problem enables one to find the stresses and displacements developed inside the body due to the propagation of waves by the action of a normal force, when the body is initially stressed or unstressed.

It may be noted from the numerical results that the initial stress system simultaneously increases and decreases the development of stresses inside the body. Further, it is seen that the development of normal incremental stresses, in particular, is more rapid with the higher initial stresses. Moreover, under the pre-stressed condition of the body, the stress development decreases with the increase of the velocity of the wave, i.e. the initial stress field influences the

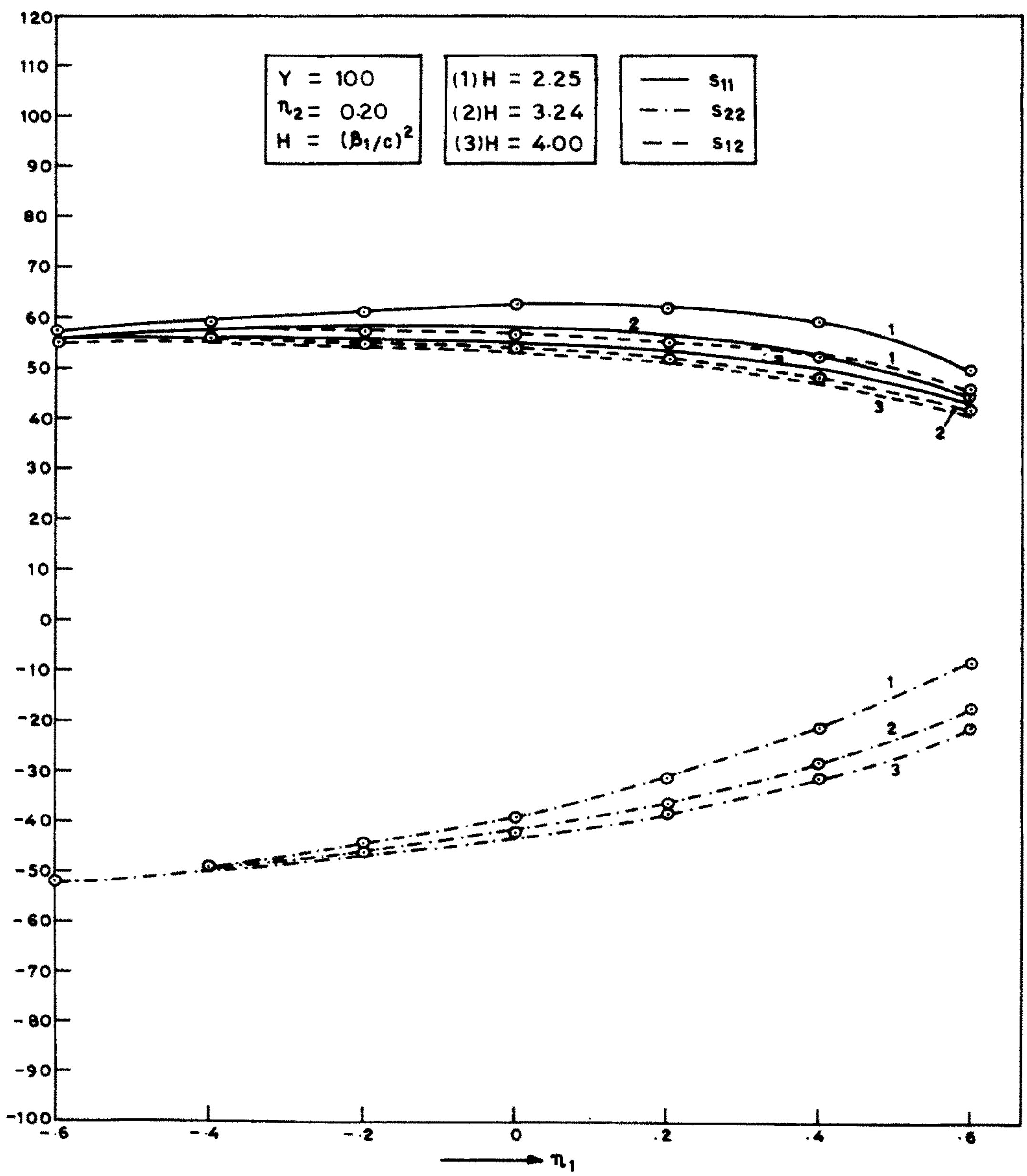

Fig. 6. Incremental stresses versus η_1 for $\eta_2 = 0.20$.

propagation of wave in such a way that the development of incremental stresses inside the solid is reduced.

From displacement analysis it is observed that the leading terms of the displacement components are complex in an initially stress-free medium whereas those are real under pre-stressed conditions. Further, the real parts of these terms contain higher values in pre-stressed state. It also may be marked that the velocity of Rayleigh wave is 0.6 times the shear wave velocity under initial compressive stress parameters (0.4 and 0.2), on the other hand it is 0.95 times the shear wave velocity in stress-free case.

Thus, it may be concluded that initial stress field has considerable effect on the propagation of waves. This type of investigation can be used to measure the strength of a body under oscillating loads.

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